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CHENG, Peng; MASUYAMA, Shigeru

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程 鵬 増山 繁
Peng CHENG *Shigeru MASUYAMA*

豊橋技術科学大学知識情報工学系
Department of Knowledge-Based Information Engineering,
Toyohashi University of Technology,
Toyohashi-shi 441, Japan

Abstract

For a probabilistic graph $(G=(V,E,s,t),p)$, where G is an undirected graph with specified source vertex s and sink vertex t $(s\neq t)$ in which each edge has independent failure probability and each vertex is assumed to be failure-free, and $p=(p(e_1),\ldots,p(e_{|E|}))$ is a vector consisting of failure probabilities $p(e_i)$'s of all edges e_i's in E, we consider the problem of computing the expected maximum number $\Gamma_{(G,p)}$ of edge-disjoint s-t paths. It has been known that this computing problem is NP-hard even if G is restricted to several classes like planar graphs, s-t out-in bitrees and s-t complete multi-stage graphs. In this paper, for a probabilistic graph $(G=(V,E,s,t),p)$, we propose a lower bound of $\Gamma_{(G,p)}$ and show the necessary and sufficient conditions by which the lower bound coincides with $\Gamma_{(G,p)}$. Furthermore, we also give a method of computing the lower bound of $\Gamma_{(G,p)}$ for a probabilistic graph $(G=(V,E,s,t),p)$.

1 Introduction

We consider a probabilistic graph $(G=(V,E,s,t),p)$, where G is an undirected graph with specified source vertex s and sink vertex t $(s\neq t)$ in which each edge has independent failure probability and each vertex is assumed to be failure-free, and $p=(p(e_1),\ldots,p(e_{|E|}))$ is a vector consisting of failure probabilities $p(e_i)$'s of all edges e_i's in E. The expected maximum number $\Gamma_{(G,p)}$ of edge-disjoint s-t paths (namely, s-t paths having no edge in common) in a probabilistic graph (G,p) is useful for network reliability analysis. Note that the problem of computing s,t-connectedness [1,3], namely, probability that there exists at least one operative s-t path, is a special case of computing $\Gamma_{(G,p)}$ in a probabilistic graph (G,p).

However, it is known that the problem of computing $\Gamma_{(G,p)}$ in a probabilistic graph (G,p) is NP-hard, even if G is restricted to several classes, e.g., planar graphs, s-t out-in bitrees and s-t complete multi-stage graphs [2]. Thus, for estimating $\Gamma_{(G,p)}$, it is interesting for us to find its lower bound in a probabilistic graph (G,p).

In this paper, we define a lower bound of $\Gamma_{(G,p)}$ using an s-t path number function of G for a probabilistic graph (G,p), and give the necessary and sufficient conditions by which this lower bound coincides with $\Gamma_{(G,p)}$ and a method of computing this lower bound. This paper is organized as follows:

Graph theoretic terminologies used throughout this paper are described in section 2. A lower bound of $\Gamma_{(G,p)}$ in a probabilistic graph (G,p) is defined in section 3. Section 4 shows the necessary and sufficient conditions by which this lower bound coincides with $\Gamma_{(G,p)}$. Furthermore, we suggest a method of computing the lower bound in section 5.

2 Preliminaries

2.1 Graph Theoretic Terminologies

A two-terminal undirected graph $G = (V, E, s, t)$ consists of a finite vertex set V and a set E of pairs of vertices, called edges, where s and t, called *source* and *sink*, respectively, are two specified distinct vertices of V. For an edge (u, v), the two vertices u and v are said to be end vertices of (u, v), and (u, v) is said to be incident to u and v.

In $G = (V, E, s, t)$, an *x-y path* π of length k from vertex x to vertex y is an alternating sequence of vertices $v_i \in V$ $(0 \le i \le k)$ and edges $(v_{i-1}, v_i) \in E$ $(1 \le i \le k)$,

$$\pi : (x =)v_0, (v_0, v_1), v_1, ..., v_{k-1}, (v_{k-1}, v_k), v_k(= y),$$

where vertices v_i's ($0 \le i \le k$) are distinct. i.e., a path denotes a simple path throughout this paper. For short, we also denote an x-y path π by

$$\pi : (x =)v_0, v_1, ..., v_{k-1}, v_k(= y).$$

The vertices $v_1, ..., v_{k-1}$ are called its internal vertices and the vertices $v_0(= s), v_k(= t)$ are called its end vertices. Let $V(\pi)$, $E(\pi)$ denote the set of all vertices and the set of all edges on an x-y path π, respectively. The set of all x-y paths in G is denoted by $P_{xy}(G)$. Paths $\pi_1, ..., \pi_r$ are called *internal vertex-disjoint paths* if they have no vertex in common except their end vertices. s-t paths $\pi_1, ..., \pi_r$ are called *edge-disjoint s-t paths* if any two of them have no edge in common, and the maximum number of edge-disjoint s-t paths in G is denoted by $\lambda_{st}(G)$.

A graph $G_1 = (V_1, E_1)$ is a subgraph of $G = (V, E, s, t)$, if $V_1 \subseteq V$ and $E_1 \subseteq E$ hold. If G_1 is a subgraph of G, other than G itself, then G_1 is a proper subgraph of G. For a subset $E' \subseteq E$, the subgraph derived from G by deleting all edges of E' is denoted by $G - E'(= (V, E - E', s, t))$. A subset $E'(\subseteq E)$ is called an *s-t edge-cutset* if $G - E'$ has no s-t path. An s-t path π is an *s-t edge-cut-path* if $E(\pi)$ is an s-t edge-cutset. An s-t edge-cutset with the minimum cardinality among s-t edge-cutsets of G is said to be *minimum*. By well-known Menger's theorem [4], $\lambda_{st}(G)$ is equal to the cardinality of a minimum s-t edge-cutset of G for any G.

2.2 Probabilistic Graph

A probabilistic graph, denoted by $(G = (V, E, s, t), p)$, or (G, p), for short, is defined as follows:
(i) $G = (V, E, s, t)$ is a two-terminal graph, where each edge e of E is in either of the following two states: failed or operative (not failed), having known independent failure probability $p(e)$, $0 \le p(e) \le 1$ (or operative probability $q(e) = 1 - p(e)$), and each vertex is assumed to be failure-free.
(ii) p is a vector consisting of all edge failure probabilities $p(e)$'s in E.

For a probabilistic graph $(G = (V, E, s, t), p)$, let a subgraph $G - U(\subseteq E)$ correspond to an event $\mathcal{E}_U$ that all edges of U are failed and all edges of $E - U$ are operative. Clearly, the probability $\rho(G - U)$ of arising a subgraph $G - U(\subseteq E)$ is computed by the following formula.

$$\rho(G - U) = \prod_{e \in U} p(e) \prod_{e \in E - U} q(e)(= 1 - p(e)).$$

Furthermore, $\sum_{U \subseteq E} \rho(G - U) = 1$ holds.

Now, we define the *expected maximum number* $\Gamma_{(G,p)}$ of edge-disjoint s-t paths in a probabilistic graph $(G = (V, E, s, t), p)$ as follows:

$$\Gamma_{(G,p)} \equiv \sum_{U \subseteq E} \lambda_{st}(G - U)\rho(G - U). \tag{1}$$

It is known that the problem of computing $\Gamma_{(G,p)}$ for a probabilistic graph (G,p) is NP-hard, even if G is restricted to several special classes like planar graphs, s-t out-in bitrees and s-t multi-stage complete graphs, etc. [2]. Thus, it is interesting for us to consider a lower bound of $\Gamma_{(G,p)}$ for estimating it.

3 A Lower Bound of $\Gamma_{(G,p)}$

We define a lower bound of the expected maximum number of edge-disjoint s-t paths in a probabilistic graph.

An *s-t path number function* f of $G=(V,E,s,t)$ is a one-to-one integral function $f : P_{st}(G) \mapsto \{1,\ldots,l\}$. The s-t path π with $f(\pi)=k$ is said to be the *s-t path of number* k, and denoted by π_k. The s-t path with the minimum number in $G-E'(\subseteq E)$ with respect to f is denoted by $\pi_{m(G-E',f)}$.

First, we give the following procedure **FEDP** to find edge-disjoint s-t paths in $G=(V,E,s,t)$.

Procedure FEDP
Input A graph $G=(V,E,s,t)$ and an s-t path number function f of G.
Output The set of edge-disjoint s-t paths $FEDP(G,f)$.

```
BEGIN
        G' := G; FEDP(G,f) := φ;
        WHILE P_st(G') ≠ φ DO
           BEGIN
              Find π_m(G',f) from P_st(G');
              FEDP(G,f) := FEDP(G,f) ∪ {π_m(G',f)};
              G' := G' − E(π_m(G',f))
           END;
        Output FEDP(G,f)
END.
```
□

It is clear that $FEDP(G,f)$ obtained by **FEDP** is a set of edge-disjoint s-t paths in G. Namely, the following formula holds.

$$|FEDP(G,f)| \leq \kappa_{st}(G), \quad for\ any\ G,\ f. \tag{2}$$

For a probabilistic graph $(G=(V,E,s,t),p)$ and an s-t path number function f of G, we now define the value $\underline{\Gamma}_{(G,f,p)}$ as follows:

$$\underline{\Gamma}_{(G,f,p)} \equiv \sum_{U\subseteq E} |FEDP(G-U,f)|\rho(G-U). \tag{3}$$

By formulas (1),(2),(3), $\underline{\Gamma}_{(G,f,p)}$ is a lower bound of $\Gamma_{(G,p)}$, namely, the following formula holds.

$$\underline{\Gamma}_{(G,f,p)} \leq \Gamma_{(G,p)}, \quad for\ any\ G,\ f,\ p.$$

4 Necessary and Sufficient Conditions

In this section, we give the necessary and sufficient conditions by which $\underline{\Gamma}_{(G,f,p)}$ coincides with $\Gamma_{(G,p)}$ in a probabilistic graph (G,p).

4.1 A Necessary and Sufficient Condition of an s-t Path Number Function

By formulas (1),(2),(3), the following Theorem 4.1 immediately holds.

Theorem 4.1. Given $(G = (V, E, s, t), p)$, then $\underline{\Gamma}_{(G,f,p)} = \Gamma_{(G,p)}$ holds iff G has an s-t path number function f satisfying the following formula.

$$|FEDP(G-U, f)| = \lambda_{st}(G-U),\ for\ any\ U \subseteq E. \tag{4}$$

□

Definition 4.1. An s-t path number function f of G is called *exact* if f satisfies formula (4). □

A graph $G = (V, E, s, t)$ is said to be *s-t k-edge-connected* if $\lambda_{st}(G) = k$ holds. A graph G is said to be π-edge-cut if π is an s-t edge-cut-path in G. A graph G is said to be *π-edge-cut s-t 2-edge-connected* if π is an s-t edge-cut-path of G and G is s-t 2-edge-connected. A π-edge-cut s-t 2-edge-connected graph $G = (V, E, s, t)$ is *minimal*, if $G - \{e\}$ for any $e \in E - E(\pi)$ is not π-edge-cut s-t 2-edge-connected. For example, the graph G shown in Fig.1 is a π-edge-cut s-t 2-edge-connected graph, where $\pi : v_0(= s), v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9(= t)$. But it is not minimal as $G - \{e\}$ is π-edge-cut s-t 2-edge-connected. Furthermore, the set of all π-edge-cut s-t 2-edge-connected subgraphs of an s-t path π of G is denoted by $\mathcal{W}(G, \pi)$. For example, in the graph G given in Fig.1, $W(G, \pi) = \{G - \{e = (u_1, u_2)\},\ G - \{(u_1, v_4), (u_2, v_5), (v_3, v_5)\}\ \}$. Clearly, the following Lemma 4.1 holds.

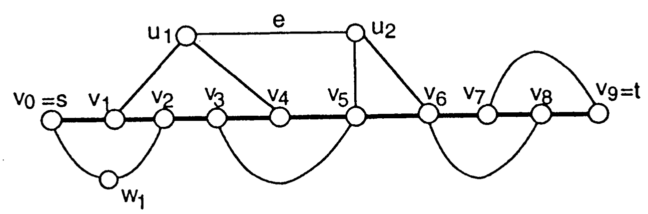

Fig.1 A π-edge-cut s-t 2-edge-connected graph.

Lemma 4.1. If $\lambda_{st}(G) \geq 2$ holds and an s-t path π of G is an s-t edge-cut-path, then $\mathcal{W}(G, \pi) \neq \phi$ holds. □

Lemma 4.2. In a graph $G = (V, E, s, t)$, if there exists an s-t path π satisfying $\mathcal{W}(G, \pi) = \phi$, then the following formula holds.

$$\lambda_{st}(G - E(\pi)) = \lambda_{st}(G) - 1.$$

Proof. Clearly, $\lambda_{st}(G - E(\pi)) \leq \lambda_{st}(G) - 1$ holds. Assume that $\lambda_{st}(G - E(\pi)) < \lambda_{st}(G) - 1$ holds. By this assumption, there exists a minimum s-t edge-cutset E^* in $G - E(\pi)$ that satisfies $|E^*| \leq \lambda_{st}(G) - 2$ by Menger's Theorem [4]. Consider graph $G - E^*$, and it is clear that all s-t paths in $G - E^*$ share at least one edge of $E(\pi)$, i.e., π is an s-t edge-cut-path of $G - E^*$. Furthermore, let E' be a minimum s-t edge-cutset of $G - E^*$. As $E' \cup E^*$ is an s-t edge-cutset of G, $|E' \cup E^*| = |E'| + |E^*| \geq \lambda_{st}(G)$ holds. By $|E^*| \leq \lambda_{st}(G) - 2$, we obtain $|E'| = \lambda_{st}(G - E^*) \geq 2$, contradicting the fact that $\mathcal{W}(G, \pi) \neq \phi$ holds by Lemma 4.1. □

We now prove the following Theorem 4.2.

Theorem 4.2. In a graph $G = (V, E, s, t)$, an s-t path number function f of G is exact iff for any $U \subseteq E$ with $P_{st}(G-U) \neq \phi$, $\mathcal{W}(G-U, \pi_{m(G-U,f)}) = \phi$ holds.

Proof. Necessity: Assume that an s-t path number function f of G is exact and that for some $U \subseteq E$ with $P_{st}(G-U) \neq \phi$, $\mathcal{W}(G-U, \pi_{m(G-U,f)}) \neq \phi$ holds. By $\mathcal{W}(G-U, \pi_{m(G-U,f)}) \neq \phi$, $G-U$ has a subgraph $G' \in \mathcal{W}(G-U, \pi_{m(G-U,f)})$. $\lambda_{st}(G') = 2$ holds by the definition of $\mathcal{W}(G-U, \pi_{m(G-U,f)})$. As $\pi_{m(G-U,f)}$ is the s-t path with the minimum number of G' and an s-t edge-cut-path of G', we have $FEDP(G', f) = \{\pi_{m(G-U,f)}\}$ by **FEDP**. Hence, $|FEDP(G', f)|(=1) < \lambda_{st}(G')(=2)$ holds, contradicting the fact that f is exact.

Sufficiency: Assume that for any $U \subseteq E$ with $P_{st}(G-U) \neq \phi$, $\mathcal{W}(G-U, \pi_{m(G-U,f)}) = \phi$ holds. Then it is easy to prove that for any $U \subseteq E$, $|FEDP(G-U, f)| = \lambda_{st}(G-U)$ holds by iteratively applying Lemma 4.2. □

4.2 A Necessary and Sufficient Condition of s-t Paths

Definition 4.2.(*Prohibitive s-t Path Set*)
Let $P(\subseteq P_{st}(G))$ be a subset of the set of all s-t paths of G. If, for each s-t path π of P, there is a π-edge-cut s-t 2-edge-connected subgraph $G_\pi \in \mathcal{W}(G, \pi)$ in G that satisfies $P_{st}(G_\pi) \subseteq P$, then P is called a ***prohibitive s-t path set.*** □

Procedure TEST
Input: A graph $G = (V, E, s, t)$.
Output: Either an s-t path number function f of G or a subset P of $P_{st}(G)$.

BEGIN
  $P := P_{st}(G)$; $i := 1$; $Q := \{\ \pi \in P_{st}(G) \mid \mathcal{W}(G, \pi) = \phi\}$;
  WHILE $Q \neq \phi$ DO
    BEGIN
      $P := P - Q$;
      REPEAT
        Select an s-t path π from Q;
        $f(\pi) := i$; $i := i + 1$; $Q := Q - \{\pi\}$
      UNTIL $Q = \phi$;
      $Q := \{\pi \in P \mid P_{st}(G_\pi) \not\subseteq P,\ for\ all\ G_\pi \in \mathcal{W}(G, \pi)\}$
    END;
  IF $P = \phi$ THEN output f ELSE output P
END. □

Clearly, the following Lemma 4.3 holds by Definitions 4.1 and 4.2.

Lemma 4.3. If **TEST** outputs an s-t path number function f of G, then f is exact, when a graph $G = (V, E, s, t)$ is input. If **TEST** outputs a subset P of $P_{st}(G)$, then P is a prohibitive s-t path set, when a graph $G = (V, E, s, t)$ is input. □

If there is a prohibitive s-t path set $P(\subseteq P_{st}(G))$ where $G = (V, E, s, t)$, then there does not exist any exact s-t path number function f. Otherwise, if G has an exact s-t path number function f, and suppose π_m be the s-t path of the minimum number with respect to f among P. By Definition 4.2, there is $G_{\pi_m} \in \mathcal{W}(G, \pi_m)$ in G that satisfies $P_{st}(G_{\pi_m}) \subseteq P$. Thus, π_m is also the s-t path of the minimum number with respect to f in G_{π_m}. Therefore, by **FEDP**, $FEDP(G_{\pi_m}, f) = 1 < \lambda_{st}(G_{\pi_m}) = 2$ holds. This leads to a contradiction that f is an exact s-t path number function of G. Hence, by Theorem 4.2 and Lemma 4.3, the following Theorem 4.3 holds.

Theorem 4.3. In a graph $G = (V, E, s, t)$, G has an exact s-t path number function iff it contains no prohibitive s-t path set as its s-t path subset. □

4.3 Characterization of Graph Having a Prohibitive s-t Path Set

A graph is connected if there is a path connecting each pair of vertices and otherwise disconnected. A connected component of G is a maximal connected subgraph, which is simply called a component. If there exist vertices x and y, $x \neq v$ and $y \neq v$ such that all the paths connecting x and y have v as an internal vertex, then v is an *articulation vertex*. A two-terminal connected graph is said to be *s,t non-separable* if its subgraph obtained by removing s, t is connected. In the following discussion, we assume that G is an s,t non-separable two-terminal connected graph, unless otherwise specified.

Definition 4.3. (*s-t 2-edge-connected Articulation Vertex*)
A vertex v is said to be an *s-t 2-edge-connected articulation vertex* of G, if v is an s-t articulation vertex of G and there exist both two edge-disjoint s-v paths and two edge-disjoint v-t paths in G. □

For example, in the graph illustrated in Fig.2(a), vertices u, v, w are s-t 2-edge-connected articulation vertices of G.

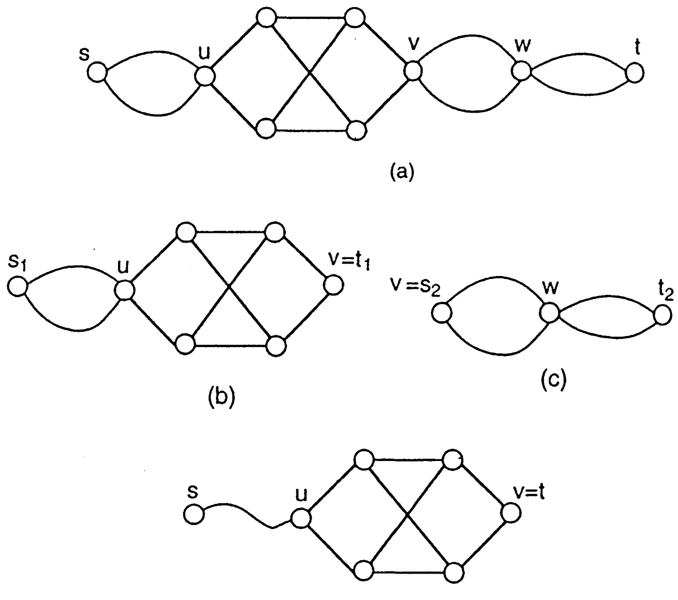

Fig.2 An illustration of separation of G at an s-t 2-edge-connected articulation vertex.

Definition 4.4. (*Separation of G at an s-t 2-edge-connected Articulation Vertex*)

Assume that G has an s-t 2-edge-connected articulation vertex v. The following sequence of operations is said to be *separation of G at an s-t 2-edge-connected articulation vertex v.*

(i) The two components C_1 and C_2 are obtained by removing v from G.

(ii) v is connected to C_1 (or C_2) with all edges (u,v)'s of G having one end vertex u in C_1 (or C_2).

(iii) Note that C_1 contains either of s,t. If C_1 contains s (or t) then let s (or t) be s_1 (or t_1) and let v be t_1 (or s_1). s_2 and t_2 are similarly defined for C_2. □

For example, the two graphs illustrated in Fig.2(b),(c) are obtained by separation of the graph given in Fig.2(a) at an s-t 2-edge-connected articulation vertex v.

Definition 4.5. (*Prohibitive Graph*)
A graph G is said to be a *prohibitive graph,* if G, or one of the graphs derived from G by separations of G at all s-t 2-edge-connected articulation vertices in G is homeomorphic to the graph shown in Fig.3. □

The two graphs illustrated in Fig.2(a),(b) are both prohibitive graphs. But the graph given in Fig.2(d), although it contains a subgraph homeomorphic to the graph shown in Fig.3, is not a prohibitive graph as the vertex u is not its s-t 2-edge-connected articulation vertex and it is not homeomorphic to the graph shown in Fig.3. It is easy to verify that for a prohibitive graph G, $P_{st}(G)$ is a prohibitive s-t path set. Thus, we immediately obtain the following Lemma 4.4.

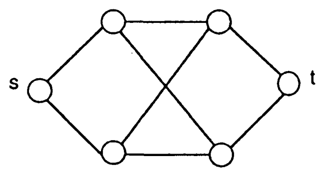

Fig.3 A prohibitive graph.

Lemma 4.4. If G contains a prohibitive graph as its subgraph, then it also has a prohibitive s-t path set as its s-t path subset. □

Now, we show that if G has a prohibitive s-t path set as its s-t path subset, then it contains a prohibitive graph as its subgraph. For our aim, we need more definitions.

Definition 4.6.(*Attachment Vertex* [5][6])
An *attachment vertex* of a subgraph G_1 in G is a vertex of G_1 incident in G with some edge not belonging to G_1. □

Definition 4.7.(*Bridges* [5],[6])
Let J be a fixed subgraph of G. A subgraph G_1 of G is said to be *J-detached* in G if all its attachment vertices are in J. We define a *bridge* of J in G as any subgraph B that satisfies the following three conditions:

(i) B is not a subgraph of J.

(ii) B is J-detached in G.

(iii) No proper subgraph of B satisfies both (i) and (ii). □

Definition 4.8.(*Degenerate and Proper Bridges. Nucleus of a Bridge* [5],[6])
An edge $e = (u,v)$ of G not belonging to J but having both end vertices in J is referred to as a *degenerate bridge.*

Let G^- be the graph derived from G by deleting the vertices of J and all edges incident to them.

Let C be any component of G^-. Let B be the subgraph of G obtained from C by adjoining to it each edge of G having one end vertex in C and the other end vertex in J and adjoining also the end vertices in J of all such edges. The subgraph B satisfies the conditions (i),(ii),(iii) in Definition 4.7 and is a bridge. Such a bridge is called to be ***proper***. The component C of G^- is the ***nucleus*** of B. □

For the graph G shown in Fig.4, let J be an s-t path $\pi : v_0(= s), v_1, v_2, v_3, v_4, v_5, v_6(= t)$, then all vertices on π other than v_4 are all attachment vertices of π in G. B_1, B_2, B_3 are proper bridges of π in G and B_4 is a degenerate bridge of π in G. By Definitions 4.6,4.7, the following Lemma 4.5 obviously holds.

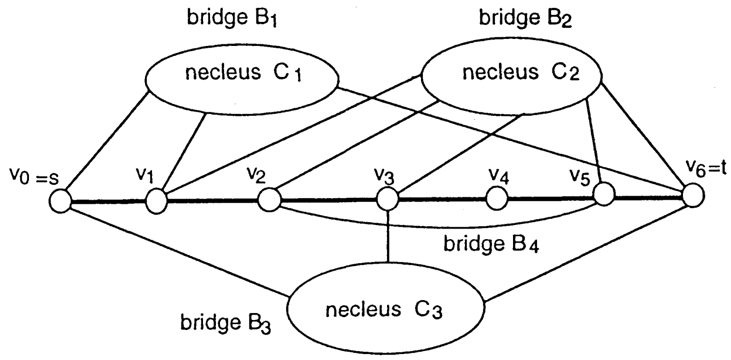

Fig.4 An illustration of attachment vertices, bridges and nuclei.

Lemma 4.5. Let π be an s-t path of G. If there is a proper bridge B of π in G, then any two vertices u, v in B are connected by a path consisting of edges and vertices only in the nucleus of B. □

Let $\gamma : v_0, v_1, ..., v_{k-1}, v_k$ be a path from v_0 to v_k of G. If $0 \le i < j \le k$, then the sequence $v_i, v_{i+1}, ..., v_{j-1}, v_j$ is a subpath of γ, and denoted by $\gamma[v_i, v_j]$.

Definition 4.9.(*Path Avoiding s-t Path π*)
Let π be an s-t path of G. For two vertices v_i, v_j in $V(\pi)$, a path between v_i and v_j consisting of edges not in $E(\pi)$ and vertices not in $V(\pi)$ except v_i, v_j is said to be ***avoiding*** π. □

For example, the path v_1, u_1, u_2, v_5 is avoiding the s-t path π in the graph G illustrated in Fig.1.

Definition 4.10. (*Order Relation with Respect to an s-t Path π*)
Let $\pi : v_0(= s), v_1, ..., v_{k-1}, v_k(= t)$ be an s-t path of G. We define an *order relation* $<_\pi$ on $V(\pi)$ with respect to π as follows: For any v_i, v_j $(0 \le i,\ j \le k)$, $v_i <_\pi v_j$ holds iff $i < j$ holds. If $v_i <_\pi v_j$, v_i (v_j) is said to be to the left (right) of v_j (v_i). □

Definition 4.11.(*Intersection Vertex of Two Paths π, α*)
Let π, α be two paths of G. A vertex v is called an ***intersection vertex*** of π, α if π and α have at least three distinct edges incident to v. The set of all intersection vertices of π, α is denoted by $V_{\pi\alpha}$. □

In the graph G given in Fig.1, for two s-t paths π and $\alpha : v_0(= s), v_1, u_1, u_2, v_6, v_7, v_9(= t)$, we have $V_{\pi\alpha} = \{v_1, v_6, v_7, v_9\}$.

Definition 4.12.(*Interlacing Subpaths*)

Suppose that G has an s-t path $\pi : v_0(=s), v_1, ..., v_{k-1}, v_k(=t)$ satisfying $\mathcal{W}(G,\pi) \neq \phi$. Let $G_\pi \in \mathcal{W}(G,\pi)$ be a minimal π-edge-cut s-t 2-edge-connected subgraph of G. Let α, β be two edge-disjoint s-t paths of G_π. Let $V_{\pi\alpha} = \{x_1, x_2, ..., x_p\}(\subseteq V(\pi))$ be the set of all intersection vertices of π, α, where $x_1 <_\pi x_2 <_\pi \cdots <_\pi x_p$. Let $V_{\pi\beta} = \{y_1, y_2, ..., y_q\}(\subseteq V(\pi))$ be the set of all intersection vertices of π, β, where $y_1 <_\pi y_2 <_\pi \cdots <_\pi y_q$. Let $V_{\pi\alpha\beta} = \{z_1, ..., z_r\}(\subseteq V(\pi))$ be the set of all vertices which π, α, β have in common, where $z_1 <_\pi z_2 <_\pi \cdots <_\pi z_r$. Subpaths $\alpha[x_i, x_{i+1}]$ of α avoiding π and $\beta[y_j, y_{j+1}]$ of β avoiding π, where either $x_i <_\pi y_j$ or $y_j <_\pi x_i$, are said to be *interlacing subpaths*, if the subpath $\pi[x_i, y_{j+1}]$ ($\pi[y_j, x_{i+1}]$) contains no vertex of $V_{\pi\alpha\beta}$ when $x_i <_\pi y_j$ ($y_j <_\pi x_i$). □

In the graph G given in Fig.1, for two edge-disjoint s-t paths;
$\alpha : v_0(=s), v_1, u_1, v_4, v_5, u_2, v_6, v_7, v_9(=t)$, $\beta : v_0(=s), w_1, v_2, v_3, v_5, v_6, v_8, v_9(=t)$,
we have $V_{\pi\alpha} = \{v_1, v_4, v_5, v_6, v_7, v_9\}$, $V_{\pi\beta} = \{v_0, v_2, v_3, v_5, v_6, v_8\}$, $V_{\pi\alpha\beta} = \{v_0, v_5, v_6, v_9\}$. And subpaths $\alpha[v_1, v_4]$ and $\beta[v_0, v_2]$ are interlacing subpaths, and $\alpha[v_7, v_9]$ and $\beta[v_6, v_8]$ are also interlacing paths. But $\alpha[v_1, v_4]$ and $\beta[v_6, v_8]$ are not interlacing subpaths as $v_5, v_6 \in V_{\pi\alpha\beta}$ are on $\pi[v_0, v_8]$.

In order to show that if graph G has a prohibitive s-t path set $P(\subseteq P_{st}(G))$, then G must contain a prohibitive graph as its subgraph, we can prove the following Lemma 4.6 and Lemma 4.7.

Lemma 4.6. Suppose that G has a prohibitive s-t path set P. Then there is an s-t path π of P whose proper bridge B in G contains two interlacing subpaths $\alpha[x_i, x_{i+1}]$ of α and $\beta[y_j, y_{j+1}]$ of β with respect to π in G_π, where G_π is a minimal π-edge-cut s-t 2-edge-connected subgraph of G, and α, β are two edge-disjoint s-t paths in G_π.

Sketch of Proof. Let P be a prohibitive s-t path set of G. We can find the s-t path π of P satisfying the following *condition I* by using the following *procedure I.*

Condition I: There is a proper bridge B of π in G suth that B contains interlacing subpaths $\alpha[x_i, x_{i+1}]$ of α and $\beta[y_j, y_{j+1}]$ of β with respect to π in G_π, where G_π is a minimal π-edge-cut s-t 2-edge-connected subgraph of G, and α, β are two edge-disjoint s-t paths in G_π.

Procedure I: Let π be an s-t path of P. Let B be a proper bridge of π in G. We do the following *Loop* iteratively.

Loop: If π satisfies *Condition I* then end. Otherwise, we can find an s-t path π' of P such that there is a bridge B' of π' in G whose nucleus contains the nuleus of B and there are more vertices in the nucleus of B' than in the nucleus of B. Let B, π be B', π', respectively.

Note that, in each loop, the nucleus of B increases at least by one vertex. Thus the loop will end in at most $|V|$ times, where V is the set of vertices in G. □

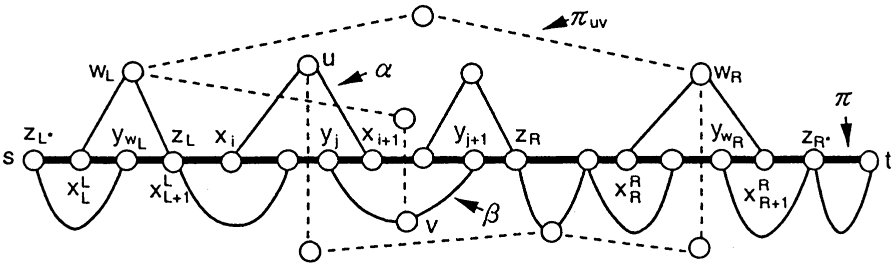

Fig.5 An illustration of the proof of Lemma 4.7.

Lemma 4.7. Suppose that G has an s-t path π satisfying $\mathcal{W}(G,\pi) \neq \phi$. Let α, β be two edge-disjoint s-t paths of $G_\pi \in \mathcal{W}(G,\pi)$. Let $V_{\pi\alpha} = \{x_1, x_2, ..., x_p\}$, $V_{\pi\beta} = \{y_1, y_2, ..., y_q\}$ and $V_{\pi\alpha\beta} = \{z_1, ..., z_r\}$

be defined as in Definition 4.12. If a bridge B of π in G contains interlacing subpaths $\alpha[x_i, x_{i+1}]$ of α and $\beta[y_j, y_{j+1}]$ of β in G_π with respect to π, then G contains a prohibitive graph as its subgraph.

Sketch of Proof. By the known conditions given in this lemma, we construct a prohibitive graph as its subgraph.

By Lemma 4.5, there is a path π_{uv} between an internal vertex u on $\alpha[x_i, x_{i+1}]$ and an internal vertex v on $\beta[y_j, y_{j+1}]$ consisting of edges and vertices only in the nucleus of bridge B, i.e., π_{uv} is vertex-disjoint path with π except u, v. See Fig.5. Thus, we can also find a prohibitive graph as subgraph of G independently of the way how the path π_{uv} is traced. □

By Theorem 4.3 and Lemmas 4.5, 4.6, 4.7, the following Theorem 4.4 holds.

Theorem 4.4. In a probabilistic graph (G,p), $\underline{\Gamma}_{(G,f,p)} = \Gamma_{(G,p)}$ holds iff G contains no prohibitive graph as its subgraph. □

5 A Method of Computing the Lower Bound

Given a probabilistic graph (G,p) and an s-t path number f of G, we show a method of computing the lower bound $\underline{\Gamma}_{(G,f,p)}$. We first wish to recall the procedure **FEDP** and the definition of $\underline{\Gamma}_{(G,f,p)}$ in section 3.

For a probabilistic graph $(G = (V, E, s, t), p)$ and an s-t path number function f of G, let $\mathcal{U}_{f,\pi_i}$ denote the set of all $U \subseteq E$ for which s-t path π_i is selected as a member of edge-disjoint s-t paths $FEDP(G-U, f)$. Let $p(\mathcal{E}_U)$ be the probability of the event $\mathcal{E}_U$ that all edges of U are failed and all edges of $E-U$ are operative, and $p(\mathcal{E}_{f,\pi_i})$ is the probability of the event that at least one event $\mathcal{E}_U$, for all $U \in \mathcal{U}_{f,\pi_i}$, arises in (G,p). Thus, we have

$$\begin{aligned}
\underline{\Gamma}_{(G,f,p)} &= \sum_{U \subseteq E} |FEDP(G-U,f)| \rho(G-U) \\
&= \sum_{i=1}^{|P_{st}(G)|} \sum_{U \in \mathcal{U}_{f,\pi_i}} \rho(G-U) \\
&= \sum_{i=1}^{|P_{st}(G)|} \sum_{U \in \mathcal{U}_{f,\pi_i}} p(\mathcal{E}_U) \\
&= \sum_{i=1}^{|P_{st}(G)|} p(\mathcal{E}_{f,\pi_i}).
\end{aligned} \tag{5}$$

We can compute the lower bound $\underline{\Gamma}_{(G,f,p)}$ by formula (5) instead of formula (3).

6 Concluding Remarks

For a probabilistic graph, we proposed a lower bound for estimating the expected maximum number of edge-disjoint s-t paths. The necessary and sufficient conditions with respect to both s-t path number function and graph construction, where this lower bound coincides with the expected maximum number of edge-disjoint s-t paths, are clarified. A method of computing this lower bound is also given, although by this computing method the lower bound does not seem to be efficiently computed for a general probabilistic graph.

However, for a probabilistic one-layered s-t graph, (a two-terminal graph where the subgraph obtained by deleting its s, t is exactly a simple path. Fig.6 illustrates an example of one-layered s-t graph.) as it satisfies the necessary and sufficient conditions and the number of all its s-t paths is a polynomial function in the number of its vertices, the lower bound based on its exact s-t path number function can efficiently be computed by the computing method shown in section 5, i.e., the expected maximum number of edge-disjoint s-t paths in a probabilistic one-layered s-t graph can efficiently be computed. Detailed description of these proofs is lengthy and to be reported elsewhere.

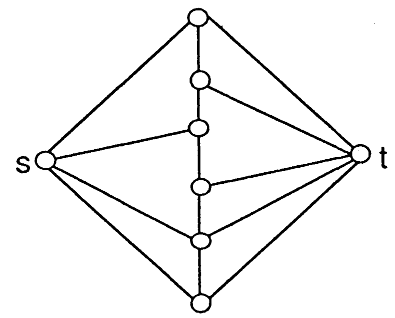

Fig.6 A one-layered s-t graph.

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